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THE RESTRICTION MAP IN COHOMOLOGY OF FINITE 2-GROUPS

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In this paper we study the mod-2 cohomology rings of finite 2-groups. More precisely we want to estimate the nilpotence degree of an ideal in $H^*(G,\mathbb{Z}_2)$ which consists of all cohomology classes whose restrictions to all proper subgroups of G are trivial. We estimated this number for 2-groups which are generated by two elements and for groups whose Frattini subgroup $\Phi(G)$ is central and is not elementary abelian.

1. Introduction

Let G be a finite 2-group. Denote by $H^*(G)$ the mod-2 cohomology ring of (the classifying space) the group G.

Definition. An element $u\in H^*(G)$ is called *essential* if $\operatorname{res}(u)=0$ for all proper subgroups of G.

For a group G which is not elementary abelian we denote by $\operatorname{Ess}(G)$ the ideal of $H^*(G)$ which consist of all essential elements. Thus

$$\operatorname{Ess}(G)=\ker\{\operatorname{res}: H^*(G)\to \textstyle\prod H^*(M)\},$$

where the product is taken over all maximal proper subgroups of G. According to Quillen's theorem [2,4] the ideal $\operatorname{Ess}(G)$ is nilpotent. Unfortunately, his proof does not provide any estimation of its nilpotence degree. Let $\operatorname{hil}(G)$ denote the nilpotence degree of the ideal $\operatorname{Ess}(G)$. Let us note first the following simple fact:

Proposition. *If G is a direct product of groups $G_1,G_2,\dots,G_n$ then* $\operatorname{hil}(G)\le \min(\operatorname{hil}(G_1),\dots,\operatorname{hil}(G_n))$. □

In this paper we prove that in certain cases $\operatorname{hil}(G)\le 2$. Note that the last proposition implies it for abelian groups. From [3, Theorem 5.10] it follows that $\operatorname{hil}(G)=1$ for groups G which are extensions of $\mathbb{Z}_2$ by an elementary abelian group V (e.g.

extra-special 2-groups) except in two cases: $G = Q_8$ and $G = \mathbb{Z}_4$ when $\mathrm{hil}(G) = 2$. Rusin in [5] and in [6] calculated the mod-2 cohomology rings of all metacyclic groups and of all groups of order 32. From his results one can deduce that $\mathrm{hil}(G) \leq 2$ for those groups.

On the other hand for an arbitrary not elementary abelian 2-group G there is an important element $e_G \in \mathrm{Ess}(G)$ defined as the product of all nonzero 1-dimensional cohomology classes. Serre [7] showed that for this particular element, but an arbitrary group, we have $e_G^2 = 0$.

Definition. We say that the group G has the *cohomological length* n ($\mathrm{chl}(G) = n$), if n is the smallest positive integer for which there exists nonzero 1-dimensional cohomology classes $x_1, x_2, \dots, x_n$ such that $x_1 \cdot x_2 \cdot \dots \cdot x_n = 0$ in $H^n(G)$.

From Serre's results follows that for an arbitrary (not elementary abelian) 2-group its cohomological length is bounded by $2 \cdot (2^m - 1)$, where $m = \dim H^1(G)$. In the next section we prove that $\mathrm{hil}(G) \leq \mathrm{chl}(G)$. This simple observation will be fundamental for the proof of two of our main results:

Proposition A. *Let G be a 2-group generated by two elements which is not elementary abelian. Then* $\mathrm{hil}(G) \leq 2$.

Proposition B. *Let G be a 2-group satisfying the following conditions*:

(i) $\Phi(G) \subseteq Z(G)$.

(ii) *$\Phi(G)$ is not an elementary abelian subgroup of G.*

Then $\mathrm{hil}(G) \leq 2$.

Unfortunately we can not extend these results to all 2-groups. We can only ask the following question:

Question. Is $\mathrm{hil}(G) \leq 2$ for an arbitrary 2-group G?

2. Proofs

Throughout this section we denote by V an elementary abelian 2-group. We identify its mod-2 cohomology ring with the symmetric algebra $S(V^{\#})$ where $V^{\#}$ is the dual (as a vector space) of V. For convenience we set $\mathrm{hil}(V) = \infty$. All group extensions are central.

Proposition 1. *If $H \subseteq G$ is a subgroup of G of index two then the sequence*

$$H^i(G) \xrightarrow{\cdot x} H^{i+1}(G) \xrightarrow{\mathrm{res}} H^{i+1}(H),$$

is exact for all i, where $x \in H^1(G)$ is a homomorphism $G \to \mathbb{Z}_2$ such that $\ker(x) = H$.

Proof. It is enough to apply the Gysin sequence to the bundle of 0-spheres $\xi : BH \to BG$, induced by inclusion $H \to G$. $\square$

The proposition implies that $\ker(\mathrm{res}_{G\to H})$ is a principal ideal generated by the element x such that $\ker x = H$. Hence $\mathrm{Ess}(G) = \bigcap \{(x): x \in H^1(G)\}$.

Corollary 2. *Let G be an arbitrary group. Then* $\mathrm{hil}(G) \leq \mathrm{chl}(G)$.

Proof. Let $n = \mathrm{chl}(G)$. Thus there exist nonzero 1-dimensional classes $x_1, x_2, \dots, x_n$ such that $x_1 \cdot x_2 \cdot \dots \cdot x_n = 0$. Proposition 1 implies that every element $u \in \mathrm{Ess}(G)$ is divisible by an arbitrary 1-dimensional cohomology class. In particular for any sequence $u_1, \dots, u_n \in \mathrm{Ess}(G)$ we have $u_1 \cdot u_2 \cdot \dots \cdot u_n = x_1 \cdot x_2 \cdot \dots \cdot x_n \cdot w = 0$, for some $w \in H^*(G)$, hence $(\mathrm{Ess}(G))^n = 0$. $\square$

Lemma 3. *Let $\pi : G \to \bar{G}$, be an epimorphism. Then* $\mathrm{chl}(G) \leq \mathrm{chl}(\bar{G})$.

Proof. Let $n = \mathrm{chl}(\bar{G})$ and let $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ be nonzero 1-dimensional cohomology classes such that $\bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_n = 0$ in $H^n(\bar{G})$. For $i = 1, \dots, n$ we define $x_i = \pi^*(\bar{x}_i)$. The induced map π^* is monic on the first cohomology group hence these elements are nontrivial. π is a group homomorphism hence π^* is a ring homomorphism so that $x_1 \cdot x_2 \cdot \dots \cdot x_n = 0$ in $H^n(G)$. $\square$

Now we recall some facts from group theory.

Lemma 4. *Let G be an n-generator group of order 2^k. Then for any $l \leq k$ there exist an epimorphism $\pi : G \to \bar{G}$, onto a group $\bar{G}$ of order 2^l. Moreover if $n \leq l \leq k$ one can find such $\bar{G}$ which also is an n-generator group.*

Proof. Proceeding by induction with respect to n we may assume that $l = k - 1$. Let $C \subseteq \Phi(G) \cap Z(G)$ be a cyclic subgroup of order 2. Clearly the projection $\pi : G \twoheadrightarrow \bar{G} = G/C$ satisfies the required conditions. $\square$

Proposition 5. *Let $\mathbb{Z}_2 \rightarrowtail G \twoheadrightarrow \bar{G}$ be an extension defined by a nontrivial cocycle $Q \in H^2(\bar{G})$ which is a product of 1-dimensional classes. Then* $\mathrm{chl}(G) = 2$.

Proof. Let $Q = u \cdot u'$ where $u, u' \in H^1(\bar{G})$. Thus $\pi^*(u), \pi^*(u') \neq 0$ and $\pi^*(u) \cdot \pi^*(u') = \pi^*(Q)$. To see that $\pi^*(Q) = 0$ in $H^2(G)$ we consider the following pull-back diagram:

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[r, tail] \arrow[d, equal] & \tilde{G} \arrow[r, two heads] \arrow[d] & G \arrow[d, "\pi"] \\
\mathbb{Z}_2 \arrow[r, tail] & G \arrow[r, two heads, "\pi"] & \bar{G}.
\end{tikzcd}$$

The top extension is defined by $\pi^*(Q)$ and it splits. $\square$

Proof of Proposition A. A 2-generator group which is not elementary abelian has at least 8 elements. There are three 2-generator groups of order 8: the product of two cyclic groups $\mathbb{Z}_2 \times \mathbb{Z}_4$, the dihedral group D_8 and the quaternion group Q_8. Their cohomology rings are as follows (cf. [5]):

$$\begin{aligned} H^*(\mathbb{Z}_2 \times \mathbb{Z}_4) &= \mathbb{Z}_2[x, y, z]/(x^2), \\ H^*(D_8) &= \mathbb{Z}_2[x, y, z]/(xy), \\ H^*(Q_8) &= \mathbb{Z}_2[x, y, t]/(x^2 + xy + y^2, x^2y + xy^2), \end{aligned}$$

where $\deg(x) = \deg(y) = 1$, $\deg(z) = 2$ and $\deg(t) = 4$. It is clear that the cohomological length of the first two groups is 2. Unfortunately $\mathrm{chl}(Q_8) > 2$, but it is straightforward to show that $\mathrm{Ess}(Q_8) = (x^2, xy, y^2)$. Thus $(\mathrm{Ess}(Q_8))^2 = 0$ and $\mathrm{hil}(Q_8) = 2$.

Let G be an arbitrary 2-generator group. By Lemma 4, there is an epimorphism $\pi : G \to \bar{G}$, where $|\bar{G}| = 8$. If $\bar{G} \cong D_8$ or $\bar{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ then Lemma 3 and Corollary 2 imply that $\mathrm{hil}(G) \leq 2$.

Suppose now that $|G| > 8$ and that $\bar{G} \cong Q_8$. In this case there is an epimorphism $G \to G'$, where G' is nontrivial extension of $\mathbb{Z}_2$ by Q_8, with a cocycle $Q \in H^2(Q_8)$. Every element of $H^2(Q_8)$ is decomposable into a product of 1-dimensional classes, thus, by Proposition 5, $\mathrm{chl}(G') = 2$. Now Lemma 3 and Corollary 2 imply that $\mathrm{hil}(G) \leq 2$. $\square$

Note that the same proof shows that if an arbitrary 2-group G admits an epimorphism $G \to \bar{G}$, where $\bar{G}$ is a 2-generator group of order 16, then $\mathrm{hil}(G) \leq 2$.

From Rusin's calculation one can deduce that among 3-generator groups of order 64 there is only single one, denoted by G', for which $\mathrm{chl}(G') \geq 3$. The group G' is an extension of $\mathbb{Z}_2$ by $\Gamma_4 d$ (for definition cf. [1]) defined by the cocycle $(x + y + z)^2 + xz$, where x, y, z are generators of $H^1(\Gamma_4 d)$ (in the notation of [6]). It is possible to show that $\mathrm{chl}(G') = 3$.

Proof of Proposition B. We assume that $\Phi(G) \subseteq Z(G)$ and $\Phi(G)$ is not elementary abelian. Thus there is a subgroup $K \subseteq \Phi(G)$ such that $\Phi(G)/K \cong \mathbb{Z}_4$ and K is central in G. Let G' be the quotient group G/K. Its Frattini subgroup is isomorphic to $\mathbb{Z}_4$ and $G'/\Phi(G') \cong G/\Phi(G)$ is an elementary abelian group denoted by V. By Lemma 3 and Corollary 2, to conclude the proof it is enough to show that $\mathrm{chl}(G') = 2$. For this purpose we consider the following diagram:

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r, equal] \arrow[d, tail] & \mathbb{Z}_2 \arrow[d, tail] & \\ \mathbb{Z}_4 \arrow[r, tail] \arrow[d, "\pi"'] & G' \arrow[r, two heads] \arrow[d] & V \arrow[d, equal] \\ \mathbb{Z}_2 \arrow[r, tail] & G'/\mathbb{Z}_2 \arrow[r, two heads] & V. \end{tikzcd}$$

We will prove that $\mathrm{chl}(G'/\mathbb{Z}_2)=2$. The bottom extension is defined by a cocycle $\pi_*(Q)\in H^2(V,\mathbb{Z}_2)$, while the cocycle $Q\in H^2(V,\mathbb{Z}_4)$ defines the middle one. Both extensions are clearly nontrivial. The Bockstein sequence corresponding to the short exact sequence of coefficients $\mathbb{Z}_2 \rightarrowtail \mathbb{Z}_4 \overset{\pi}{\twoheadrightarrow} \mathbb{Z}_2$ implies that:

$$\mathrm{im}\{\pi_*: H^2(V,\mathbb{Z}_4)\to H^2(V,\mathbb{Z}_2)\} = \ker\{\beta: H^2(V,\mathbb{Z}_2)\to H^3(V,\mathbb{Z}_2)\}.$$

The Bockstein homomorphism β coincides with first Steenrod square Sq^1. Thus from the Cartan formula we obtain that

$$\ker\{\mathrm{Sq}^1: H^2(V,\mathbb{Z}_2)\to H^3(V,\mathbb{Z}_2)\} = \{x^2 : x\in H^1(V)\}.$$

Thus in particular $\pi^*(Q)=x^2$ for some $x\in H^1(V)$. Now Proposition 5 and Lemma 3 imply that $\mathrm{chl}(G)\le \mathrm{chl}(G')\le \mathrm{chl}(G'/\mathbb{Z}_2)=2$. $\square$

References

[1] M. Hall and J.K. Senior, The Groups of Order 2^n ($n\le 6$) (Macmillan, New York, 1964).

[2] D. Quillen, A cohomological criterion for p-nilpotence, J. Pure Appl. Algebra 1(4) (1971) 361–372.

[3] D. Quillen, The mod-2 cohomology rings of extra-special 2-groups and the spinor groups, Math. Ann. 194 (1971) 197–212.

[4] D. Quillen and B.B. Venkov, Cohomology of finite groups and elementary abelian subgroups, Topology 11 (1972) 317–318.

[5] D.J. Rusin, The mod-2 cohomology of metacyclic 2-groups, J. Pure Appl. Algebra 44 (1987) 315–327.

[6] D.J. Rusin, The cohomology of the groups of order 32, Math. Comp. 53 (187) (1989).

[7] J.P. Serre, Une relation dans la cohomologie des p-groupes, C.R. Acad. Sci. Paris, Sér. I 304 (20) (1987).